LINEAR ALGEBRA
CORE-12

1 mark questions

1. Define a vector space.
2. Give an example of a subspace of R^3.
3. State the algebraic properties of subspaces.
4. What is the quotient space of a vector space?
5. Define linear combination of vectors.
6. What is the linear span of a set of vectors?
7. Explain linear independence of vectors.
8. Define basis and dimension of a vector space.
9. Calculate the dimension of a subspace spanned by given vectors.
10. Define a linear transformation.
11. What is the null space of a linear transformation?
12. Explain the rank and nullity of a linear transformation.
13. Define a vector space V over a field F.
14. Give an example of a subspace $U \subseteq V$.
15. State the closure properties of subspaces.
16. Define the quotient space V/W, where W is a subspace of V.
17. Write the expression for a linear combination of vectors.
18. Define the linear span of a set of vectors {v1, v2, ... , vn}.
19. State the condition for linear independence of vectors {v1, v2, ... , vn}.
20. Define a basis B for a vector space V.
21. Calculate the dimension of a vector space dim(V).
22. Define a linear transformation $T: V \rightarrow W$.
23. Write the null space of a linear transformation: N(T).
24. State the rank-nullity theorem: rank(T) + nullity(T) = dim(V).
25. How do you represent a linear transformation using matrices?
26. State the properties of algebra of linear transformations.
27. Define isomorphism between vector spaces.
28. State an isomorphism theorem.
29. What is the dual space of a vector space?
30. Define dual basis and double dual.
31. Explain the transpose of a linear transformation and its matrix in the dual basis.
32. What are annihilators of subsets of a vector space?
33. Briefly explain the concept of fields.
34. Write the matrix representation [T] of a linear transformation T.
35. State the properties of algebra of linear transformations.
36. Define an isomorphism between vector spaces.
37. State the first isomorphism theorem for vector spaces.
38. Define the dual space V^* of a vector space V.
39. Write the expression for the dual basis β^* of a basis β.
40. Write the transpose of a linear transformation T.
41. Define the annihilator of a subset S of V: Ann(S).
42. Define a field F.
43. Define eigenspace of a linear operator.
44. What does it mean for a linear operator to be diagonalizable?
45. State the Cayley-Hamilton theorem.

46. Explain the concept of an invariant subspace.
47. Define the minimal polynomial of a linear operator.
48. What is an inner product space?
49. State the Gram-Schmidt orthogonalization process.
50. Define the eigenspace $E(\lambda)$ of a linear operator T.
51. State the condition for a linear operator to be diagonalizable.
52. State Cayley-Hamilton theorem for a linear operator T.
53. Define an invariant subspace under a linear operator T.
54. Write the minimal polynomial $\mu T(x)$ of a linear operator T.
55. Define an inner product space V with inner product $\langle\cdot,\cdot\rangle$.
56. State the Gram-Schmidt orthogonalization process.
57. Define orthogonal complements of subspaces.
58. State Bessel's inequality.
59. What is the adjoint of a linear operator?
60. Explain the concept of least squares approximation.
61. Define normal and self-adjoint operators.
62. What is an orthogonal projection?
63. State the spectral theorem.
64. Define the orthogonal complement $U\perp$ of a subspace U.
65. Write Bessel's inequality for an inner product space.
66. Define the adjoint T* of a linear operator T.
67. Write the expression for a least squares approximation solution.
68. Define a normal linear operator.
69. Define a self-adjoint linear operator.
70. Define an orthogonal projection operator P.
71. State the spectral theorem for self-adjoint operators.

2/3 marks questions

1. Prove that the intersection of two subspaces is also a subspace: $U \cap W = ?$
2. Show that the set of all 2x2 symmetric matrices forms a subspace of the vector space of 2x2 matrices.
3. Determine if the vectors {v1, v2, v3} are linearly independent. Justify your answer.
4. Given a linear transformation T: $\mathbb{R}$^3 → $\mathbb{R}$^2, find its matrix representation [T].
5. Prove that if U and W are subspaces of V, then $U \cap W$ is also a subspace of V.
6. If a set of vectors {v1, v2, v3} spans a vector space V, can we remove one vector and still have a spanning set? Explain.
7. Calculate the dimension of the null space of the matrix A = [1 2 3; 2 4 6; 0 1 2].
8. Given two linear transformations T: V → W and U: W → X, find the matrix representation of the composition $U \circ T$.
9. Prove that the sum of the dimensions of a subspace and its orthogonal complement is equal to the dimension of the whole space.
10. Show that the set of all 2x2 invertible matrices forms a group under matrix multiplication.
11. Prove that an isomorphism preserves linear independence of vectors.

12. Given an inner product space V, demonstrate that the map T: V → V* defined by T(v)(f) = f(v) is an isomorphism.
13. Find the dual basis β* for the basis β = {v1, v2, v3} of a vector space V.
14. If T: V → W is an isomorphism, what can you say about the dimensions of V and W?
15. Prove that if T is a linear transformation and dim(V) = dim(W), then T is injective if and only if it is surjective.
16. Derive the expression for the change of coordinate matrix P when transitioning between bases β and γ.
17. Given a subspace W of a vector space V, find the annihilator of W: Ann(W).
18. Determine whether the linear operator T is diagonalizable: T(x, y) = (3x + y, x + 3y).
19. Prove that eigenvectors corresponding to distinct eigenvalues of a linear operator are linearly independent.
20. Show that the matrix A = [2 1; 1 2] is diagonalizable.
21. Given a linear operator T, find its eigenspace E(λ) corresponding to the eigenvalue λ.
22. Prove or disprove: The eigenvalues of a self-adjoint linear operator are always real.
23. Verify the Cauchy-Schwarz inequality for vectors u and v in an inner product space.
24. Show that an inner product space is a normed vector space, and derive the properties of the induced norm.
25. Prove that the orthogonal complement of the null space of a matrix A is the row space of its transpose: (N(A))^⊥ = R(A^T).
26. Show that the adjoint of the adjoint of a linear operator is the operator itself: (T*)* = T.
27. Prove that the sum of orthogonal projections onto mutually orthogonal subspaces is the same as the projection onto their direct sum.
28. Given a self-adjoint linear operator T, show that its eigenvalues are real and its eigenvectors are orthogonal.
29. Derive the expression for the least squares solution x* of the system Ax = b.
30. Prove that if T is a normal operator, then T and its adjoint T* commute: TT* = T*T.
31. Show that every self-adjoint operator is normal, but not every normal operator is self-adjoint.

<u>6/7 marks questions</u>

1. Prove that the union of two subspaces is not necessarily a subspace. Provide a counterexample.
2. Consider a vector space V over a field F. Define and prove the properties of a direct sum of subspaces U and W of V.
3. Given a set of vectors {v1, v2, ... , vn}, determine whether they form a basis for the vector space V. If not, find a basis for the span of the vectors.

4. Let T: V → W be a linear transformation. Prove that the null space of T, denoted as N(T), is a subspace of V.
5. Show that if T: V → W is an isomorphism, then its inverse T^(-1): W → V is also an isomorphism.
6. Prove that the union of two subspaces is a subspace if and only if one subspace is contained within the other.
7. Consider a set of vectors S = {v1, v2, ... , vn}. Prove that the span of S is the smallest subspace of V containing all vectors in S.
8. Given a linear transformation T: V → W, define and prove the rank-nullity theorem using the concepts of rank, nullity, and dimension.
9. Let T: V → W be a linear transformation. Prove that the null space of T is a subspace of V, and the range of T is a subspace of W.
10. Prove that a linear transformation T: V → W is injective (one-to-one) if and only if its null space N(T) = {0}.
11. Prove that the composition of two isomorphisms between vector spaces is itself an isomorphism.
12. Given a linear transformation T: V → W and its matrix representation [T] with respect to bases β and γ, derive the matrix representation of T^(-1).
13. Show that if V is finite-dimensional, then the dual space V* is also finite-dimensional, and dim(V*) = dim(V).
14. Given a linear operator T and its matrix representation [T] with respect to an orthogonal basis, prove that the adjoint T* has a diagonal matrix representation.
15. Prove that the product of two self-adjoint operators is self-adjoint if and only if they commute.
16. Given a linear transformation T: V → W and its matrix representation [T] with respect to bases β and γ, prove that [T]γ = P^{-1} [T]β P, where P is the change of basis matrix.
17. Show that if T: V → W is an isomorphism, its matrix representation [T]βγ with respect to bases β and γ is invertible.
18. Prove that the dual space V* is also a vector space and has the same dimension as V.
19. Given a linear transformation T: V → W and its adjoint T*: W* → V*, prove that (S ∘ T)* = T* ∘ S* for any linear transformation S: W → X.
20. Using the properties of orthogonal projections, prove that every self-adjoint operator is diagonalizable and its eigenvalues are real.
21. Prove that if T is a normal linear operator, then T and its adjoint T* commute.
22. Given a linear operator T and its eigenvectors {v1, v2, ... , vn} corresponding to distinct eigenvalues, prove that they are linearly independent.
23. Show that every inner product space has an orthonormal basis, and use this to establish the existence of a matrix representation of a linear operator with respect to an orthonormal basis.

24. Prove that if A and B are positive definite matrices, then their sum A + B is also positive definite.
25. Using the spectral theorem, prove that a self-adjoint operator T can be orthogonally diagonalized.
26. Consider a linear operator T on a finite-dimensional inner product space V. Prove that T is normal if and only if it can be diagonalized by a unitary matrix.
27. Prove that if a linear operator T has distinct eigenvalues, then its eigenvectors corresponding to distinct eigenvalues are linearly independent.
28. Given an inner product space V and a set of linearly independent vectors {v1, v2, ... , vn}, construct an orthonormal basis for the subspace spanned by these vectors using the Gram-Schmidt process.
29. Show that the inner product space of continuous functions [a, b] is complete, thus forming a Hilbert space.
30. Prove the parallelogram law for normed vector spaces: $||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$.
31. Prove that the orthogonal projection operator P onto a subspace U is self-adjoint and idempotent ($P^2 = P$).
32. Given an orthogonal basis β = {v1, v2, ... , vn} of an inner product space V, derive the expression for the orthogonal projection of a vector v onto the subspace spanned by β.
33. Show that if T is a self-adjoint operator on a finite-dimensional inner product space, then all eigenvalues of T are real.
34. Prove that an orthogonal matrix is invertible and its inverse is also orthogonal.
35. Use the spectral theorem to prove that a normal operator T can be unitarily diagonalized.
36. Prove that the orthogonal complement $U^\perp$ of a subspace U is indeed a subspace and that $V = U \oplus U^\perp$.
37. Given an inner product space V, show that the orthogonal projection operator P onto a closed subspace U is self-adjoint and idempotent ($P^2 = P$).
38. Prove that for a self-adjoint operator T on a finite-dimensional inner product space V, the eigenvectors corresponding to distinct eigenvalues are orthogonal.
39. Given a linear operator T on a finite-dimensional inner product space V, prove that T is self-adjoint if and only if its matrix representation [T] with respect to an orthonormal basis is a Hermitian matrix.
40. Prove the spectral theorem for self-adjoint operators: Every self-adjoint operator on a finite-dimensional inner product space is diagonalizable and its eigenvalues are real.